

General Solutions of Braneworlds under the Schwarzschild Ansatz

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General solutions for braneworld dynamics coupled with the bulk Einstein equation are derived under the Schwarzschild ansatz. They relate the brane metric to the exterior configurations, and establish fine tuning conditions for the braneworlds to reproduce successes of the Einstein gravity.

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Einstein gravity is successful in (i) explaining the Newton's law of universal gravitation for moderate distances, and in (ii) predicting the observed light deflections and planetary perihelion precessions due to stellar gravity. They are derived via the Schwarzschild solution of the Einstein equation based on the ansatz, (a) staticity, (b) spherical symmetry, (c) asymptotic flatness, and (d) emptiness except for the core of the sphere.

There are many plausible reasons that tempt us to take our 3+1 dimensional curved spacetime as a “braneworld” embedded in higher dimensions [1]–[35]. The braneworld theories would be expected to inherit the successes of the Einstein gravity. It is, however, not true because there exists no Einstein equation of the braneworld itself. The brane metric alone is insufficient to specify the brane configuration in the bulk. Instead, we should use the brane position variable, which obeys its own dynamics, but not the Einstein equation [3]. To reproduce the Einstein gravity on the brane, we need finely to tune exterior configurations in general. A notable exception is the “brane induced gravity” [4], [12], [20], [21], [23], [30], [31], where the brane Einstein equation is supplied by quantum “induced gravity” [36] with a composite metric. Otherwise, however, closer examinations are desired. Any entity can be physical only through its dynamics. Hence, we derive *general solutions for braneworld dynamics* coupled with the bulk Einstein equation under the Schwarzschild ansatz (a)–(d), and establish (fine tuning) conditions to reproduce the successful results of the Einstein gravity. We first illustrate them in (A) Nambu-Goto brane in asymptotically flat bulk, and extend it to (B) warped bulk, and to (C) Einstein-Hilbert brane.

Let us consider a 3+1 dimensional braneworld embedded in 4+1 dimensions. Let X^I be the bulk coordinate, and $\hat{g}_{IJ}(X^K)$ be the bulk metric at the point X^K [37]. Let the brane be located at $X^I = Y^I(x^\mu)$ in the bulk, where x^μ are parameters. The dynamical variables of the system are $Y^I(x^\mu)$, $\hat{g}_{IJ}(X^K)$ and matter fields, but not the brane metric $g_{\mu\nu} = Y^I_{,\mu} Y^J_{,\nu} \hat{g}_{IJ}(Y^K)$ [37]. Here we consider the simplest models given by the action,

$$-\int \sqrt{-g}(\lambda + \gamma R) d^4x - \int \sqrt{-\hat{g}}(\hat{\lambda} + \hat{\gamma} \hat{R}) d^5X + S_m, \quad (1)$$

where $g = \det g_{\mu\nu}$, $\hat{g} = \det \hat{g}_{IJ}$, $R = R^\mu_{\mu}$, $\hat{R} = \hat{R}^I_I$,

$R_{\mu\nu} = R^\lambda_{\mu\nu\lambda}$, $\hat{R}_{IJ} = \hat{R}^K_{IJK}$, $R^\rho_{\mu\nu\lambda}$ and \hat{R}^L_{IJK} are the brane and bulk curvature tensors written in terms of $g_{\mu\nu}$ and \hat{g}_{IJ} , respectively, S_m is the matter action, and λ , γ , $\hat{\lambda}$, and $\hat{\gamma}$ are constants. We add no artificial fine-tuning terms. The equations of motion derived from (1) are

$$\left[-\lambda g^{\mu\nu} + \gamma \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) - T^{\mu\nu} \right] Y^I_{,\mu\nu} = 0, \quad (2)$$

$$-\hat{\lambda} \hat{g}_{IJ} + \hat{\gamma} \left(\hat{R}_{IJ} - \frac{1}{2} \hat{R} \hat{g}_{IJ} \right) = \hat{T}_{IJ}, \quad (3)$$

and those for matters, where $T_{\mu\nu}$ and \hat{T}_{IJ} are the energy momentum tensors with respect to $g_{\mu\nu}$ and \hat{g}_{IJ} , respectively [37]. Eq. (2) is the Nambu-Goto+Regge-Teitelboim equation [3] for the *braneworld dynamics*, and eq. (3) is the bulk Einstein equation. The brane Einstein equation does not hold. Hence, we solve (2) and (3) at the brane, and examine their implications. (If necessary, they can be continued into the bulk in known methods [24]–[29], [35].) We first illustrate them in a simple model in (A), and discuss extensions in (B) and (C).

(A) *Nambu-Goto Brane in Asymptotically Flat Bulk* ($\hat{\gamma} \neq 0$, $\lambda \neq 0$, $\hat{\lambda} = \gamma = 0$) We introduce the Gaussian normal coordinate $(X^\mu, X^4) = (x^\mu, \xi)$ with $\xi = 0$ at the brane, and expand the bulk metric \hat{g}_{IJ} in ξ as

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} - 2\xi K_{\mu\nu} + \xi^2 C_{\mu\nu} + O(\xi^3), \\ \hat{g}_{\mu 4} &= O(\xi^3), \quad \hat{g}_{44} = -1 + O(\xi^3), \end{aligned} \quad (4)$$

where $K_{\mu\nu}$ and $C_{\mu\nu}$ are functions of x^μ . The former is the extrinsic curvature: $K_{\mu\nu} = Y^4_{,\mu\nu} = \hat{\Gamma}^4_{\mu\nu}|_{\xi=0}$ with the bulk affine connection $\hat{\Gamma}^I_{JK}$. We introduce the time, radial, polar and azimuth coordinates, $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$, respectively, according to ansatz (a) and (b), which imply that \hat{g}_{IJ} is diagonal, and \hat{g}_{00} , \hat{g}_{11} , $\hat{g}_{22} = \hat{g}_{33}/\sin^2\theta$, and \hat{g}_{44} are functions of r and ξ . We parameterize them with

$$g_{\mu\nu} = \text{diag}(f, h, r^2, r^2 \sin^2 \theta), \quad (5)$$

$$K^\mu_\nu = \text{diag}(a, b, c, c), \quad U^\mu_\nu = \text{diag}(u, v, w, w), \quad (6)$$

where $U^\mu_\nu = (\hat{R}^{\mu}_{\nu 4} - \delta^\mu_\nu \hat{R}^{4\lambda}_{\lambda 4})|_{\xi=0}$ represents the degrees of freedom of $C_{\mu\nu} = K^\lambda_\mu K_{\lambda\nu} + \hat{R}^4_{\mu\nu 4}|_{\xi=0}$, and f , h , a , b , c , u , v , and w are functions of r . Then, eq. (2) implies

$$a + b + 2c = 0, \quad (7)$$

while eq. (3) at the brane leads to

$$-(h^{-1})'/r + (1 - h^{-1})/r^2 - 2bc - c^2 + u = 0, \quad (8)$$

$$-f'/rfh + (1 - h^{-1})/r^2 - 2ac - c^2 + v = 0, \quad (9)$$

$$-((fh)^{-1/2}f')'/2(fh)^{1/2} - (f/h)'/2rf - ab - ac - bc + w = 0, \quad (10)$$

$$-((fh)^{-1/2}f')'/2(fh)^{1/2} - (f/h)'/rf + (1 - h^{-1})/r^2 - ab - 2ac - 2bc - c^2 = 0, \quad (11)$$

where ansatz (d) is used, and contributions from the brane action to \hat{T}_{IJ} are neglected. We have five equations (7)–(11) for eight variables f , h , a , b , c , u , v , and w . Accordingly, the general solution involves three arbitrary functions, for which we here choose a , c and v .

We first eliminate f , f' , f'' and b from (7), (9) and (11) to obtain the differential equation,

$$(rh^{-1})' = 1 + r^2V + \frac{2r^2(4V + rV' - A)}{h(1 + r^2V) - 3} \quad (12)$$

$$\text{with } V = v - 2ac - c^2, \quad (13)$$

$$A = 2a^2 + 4ac + 6c^2, \quad (14)$$

To solve eq. (12), we rewrite it into the form

$$Z' = P + Q/(1 + Z/r) \quad (15)$$

$$\text{with } Z = 3(h^{-1} - 1)/2 - Vr^2/2, \quad (16)$$

$$P = (A - 4V - 3V'r/2)r^2, \quad (17)$$

$$Q = (1 + Vr^2)(A - 4V - V'r)r^2/2. \quad (18)$$

A sufficient condition for existence of the unique solution to (15) with the initial value $Z = Z_{(0)}$ at $\rho \equiv 1/r = 0$ is that, in a region D in the ρ - Z plane including $(0, Z_{(0)})$,

$$P/\rho^2 + Q/\rho^2(1 + Z\rho) \text{ is continuous, and} \quad (19)$$

$$\exists M : \text{real } \forall(\rho, z_1), (\rho, z_2) \in D$$

$$|Q/\rho(1 + z_1\rho)(1 + z_2\rho)| \leq M, \quad (20)$$

which we assume here. Then, the solution to (15) is given by $Z = \lim_{n \rightarrow \infty} Z_{(n)}$ with

$$Z_{(n)} = Z_{(0)} + \int_{\infty}^r \left[P + \frac{Qr}{r + Z_{(n-1)}} \right] dr. \quad (n \geq 1) \quad (21)$$

Another convenient form of the solution would be given by series expansion in $1/r$. Eqs. (19) and (20) indicate that $Pr = -Qr = \text{finite}$ at $r = \infty$. Hence, we assume existence of (at least, asymptotic) expansions:

$$Z = Z_0 + Z_1/r + Z_2/r^2 + \dots, \quad (22)$$

$$P = P_1/r + P_2/r^2 + \dots, \quad (23)$$

$$Q = Q_1/r + Q_2/r^2 + \dots, \quad (24)$$

where Z_i , P_i and Q_i ($i = 1, 2, \dots$) are constants, and $Z_0 = Z_{(0)}$. Then, eq. (15) implies the recurrence,

$$Z_n = - \sum_{k=0}^{n-1} \frac{Z_k[(n-1)Z_{n-k-1} + 2P_{n-k}]}{2n} - \frac{\Sigma_{n+1}}{n}, \quad (25)$$

for $n \geq 1$, where $\Sigma_n = P_n + Q_n$, or, explicitly,

$$Z_1 = -Z_0P_1 - \Sigma_2, \quad (26)$$

$$Z_2 = [Z_0^2P_1 - Z_0(P_1^2 - Q_2) + P_1\Sigma_2 - \Sigma_3]/2, \quad (27)$$

$$Z_3 = -[2Z_0^3P_1 + Z_0^2(5P_1^2 + 2Q_2) + Z_0(5P_1^3 + 4P_1P_2 + 7P_1Q_2 - 2Q_3) + P_1^2\Sigma_2 - P_1\Sigma_3 + 2Q_2\Sigma_2 + 2\Sigma_4]/6, \quad (28)$$

$$\dots \quad \dots \quad \dots$$

Once the function Z of r is thus determined, the general solution of the braneworld is given by

$$h = 1/(1 + 2Z/3r + Vr^2/3), \quad (29)$$

$$f = \exp \int_{\infty}^r \frac{h(1 + Vr^2) - 1}{r} dr, \quad (30)$$

$$u = [(r/h)' - 1]/r^2 - 2ac - 3c^2, \quad (31)$$

$$w = -(u + v)/2, \quad (32)$$

$$b = -a - 2c, \quad (33)$$

with three arbitrary functions a , c and v . The braneworld dynamics (2) is crucially important. If we omitted it, and used only the bulk Einstein equation (3), we would have had the wrong solution without (33) and with

$$A = -2ab - 4ac - 4bc - 2c^2, \quad (34)$$

$$u = [(r/h)' - 1]/r^2 + 2bc + c^2, \quad (35)$$

in the places of (14) and (31), respectively, which would cause large differences in the consequences below.

Now we compare our general solution with the results (i) and (ii) of the Einstein gravity in 3+1 dimensions. The latter is given by (8) and (9) with $a = b = c = u = v = 0$, and is solved to give the Schwarzschild solution $f = h^{-1} = 1 - \mu/r$ with an arbitrary constant μ , which is determined phenomenologically. In geodesic motions of objects, the function $V_N \equiv f - 1 = -\mu/r$ serves as the Newtonian potential for gravitational forces for moderate distances, as renders a precise explanation for the Newton's law of the universal gravitation. The angles $\Delta\varphi_E$ and $\Delta\psi_E$ of light deflection and planetary perihelion precession, respectively, due to the stellar gravity are given by $\Delta\varphi_E = 2\mu/r_0$ and $\Delta\psi_E = 3\pi\mu/(1 - e^2)a$, where r_0 is the minimum of r on the light orbit, and e and a are, respectively, the eccentricity and the average distance of the planetary orbit. These predictions precisely match with the observations.

Now we turn to our general solution. The Newtonian potential $V_N \equiv f - 1$ has large arbitrariness owing to those of a , c and v . In fact, we can always choose appropriate a , b , c , u , w and v for arbitrary f and h with $R \geq 0$, while we have no solution for $R < 0$. Therefore, we cannot predict that $V_N = -\mu/r$, unlike the Einstein gravity. The singularity structures also depend on exterior configurations. The conditions (19) and (20) would be useful in investigating them.

In ordinary observations with $r \gg \mu$, the μ/r term dominates V_N . Then, we parametrize the deviations from the results of the Einstein gravity by

$$f = 1 - \mu/r + f_2/r^2 + f_3/r^3 + \dots, \quad (36)$$

$$h^{-1} = 1 - \mu/r + \varepsilon_1/r + \varepsilon_2/r^2 + \dots \quad (37)$$

with constants f_i ($i \geq 2$) and ε_i ($i \geq 1$). The asymptotic flatness implies $f, h \rightarrow 1$ and $f', h' \rightarrow 0$ as $r \rightarrow \infty$. Then, eq. (9) (with $\hat{\lambda} = 0$) implies $Vr^2 \rightarrow 0$. This and eqs. (17)–(20) indicate $Ar^2 \rightarrow 0$, and therefore (34) implies $ar, br, cr \rightarrow 0$. These and $Vr^2 \rightarrow 0$ indicate $vr^2 \rightarrow 0$. Hence, we assume existence of (at least, asymptotic) expansions of the arbitrary functions:

$$a = a_2/r^2 + a_3/r^3 + \dots, \quad (38)$$

$$c = c_2/r^2 + c_3/r^3 + \dots, \quad (39)$$

$$v = v_3/r^3 + v_4/r^4 + \dots \quad (40)$$

with constants a_i, c_i ($i \geq 2$) and v_i ($i \geq 3$). Then, the present general solution indicates

$$\varepsilon_1 = v_3, \quad (41)$$

$$f_2 = 3\mu v_3/4 - v_4 - a_2^2 - 2c_2^2, \quad (42)$$

$$\varepsilon_2 = \mu v_3/2 - v_4 - 2a_2^2 - 2a_2c_2 - 5c_2^2, \quad (43)$$

... ..

In terms of them, the angles $\Delta\varphi$ and $\Delta\psi$ of light deflection and planetary perihelion precession, respectively, due to the stellar gravity are given by, up to $O(r^3)$,

$$\Delta\varphi \approx \Delta\varphi_E(1 - \varepsilon_1/2\mu) = \Delta\varphi_E(1 - v_3/2\mu), \quad (44)$$

$$\begin{aligned} \Delta\psi &\approx \Delta\psi_E[1 - (\mu\varepsilon_1 + 2f_2)/3\mu^2] \\ &= \Delta\psi_E\left[1 - \frac{5v_3}{6\mu} + \frac{2(v_4 + a_2^2 + 2c_2^2)}{3\mu^2}\right]. \end{aligned} \quad (45)$$

Consequently, the condition to match the observations is

$$v_3 \approx 0, \quad v_4 + a_2^2 + 2c_2^2 \approx 0. \quad (46)$$

The condition for the bulk Einstein equation (8–11) at the brane to imply the brane-Einstein equation would be

$$u = 2bc + c^2, \quad v = 2ac + c^2, \quad w = ab + ac + bc. \quad (47)$$

They are, however, not all physical. Under the dynamics, (47) leads to $a^2 + 2ac + 3c^2 = 0$. Hence, the brane Einstein equation holds only when

$$a = b = c = u = v = w = 0. \quad (48)$$

(B) *Warped Bulk* ($\hat{\lambda} \neq 0, \hat{\gamma} \neq 0, \lambda \neq 0, \gamma = 0$) The dynamics (eqs. (2) and (3)) has a solution with a flat brane at $|\xi| \leq \delta$ in empty warped bulk [19]:

$$\hat{g}_{IJ} = \text{diag}(F, -F, -Fr^2, -Fr^2 \sin^2 \theta, -1), \quad (49)$$

where δ is an infinitesimal constant, and F is a smooth function of ξ such that $F'|_{\xi=0} = 0$ and $F = e^{-2k|\xi|}$

for $|\xi| \geq \delta$ with $k^2 = -\hat{\lambda}/6\hat{\gamma}$. We seek for the general solutions, where the metric \hat{g}_{IJ} approaches (49) as $r \rightarrow \infty$. Hence, $\hat{\Gamma}_{\mu\nu}^4|_{\xi=\pm\delta} \rightarrow \mp kg_{\mu\nu}$, $\hat{\Gamma}_{\mu\nu}^4|_{\xi=0} \rightarrow 0$, and $\hat{R}^{4\mu}{}_{\nu 4}|_{\xi=\pm\delta} \rightarrow k^2\delta_\nu^\mu$. The ansatz (c) is retained on the brane, but (d) is slightly violated by matter distributed in $|\xi| \leq \delta$. We assume that the brane-generating interactions are much stronger than gravity at short distances of $O(\delta)$, while their gravitations are much weaker than those by the core of the sphere. Then, $\hat{T}_{\mu\nu}$ in $|\xi| \leq \delta$ becomes independent of r , and so does $(\hat{\Gamma}_{\mu\nu}^4|_{\xi=\pm\delta} - \hat{\Gamma}_{\mu\nu}^4|_{\xi=0})$ [38], so that it is equal to its asymptotic value $\mp kg_{\mu\nu}$. We define the functions $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{u}^\pm, \tilde{v}^\pm$ and \tilde{w}^\pm of r by

$$\hat{\Gamma}_{\mu\nu}^4|_{\xi=0} = \hat{\Gamma}_{\mu\nu}^4|_{\xi=\pm\delta} \pm kg_{\mu\nu} = \text{diag}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{c}), \quad (50)$$

$$\begin{aligned} (\hat{R}^{4\mu}{}_{\nu 4} - \delta_\nu^\mu \hat{R}^{4\lambda}{}_{\lambda 4})|_{\xi=\pm\delta} \\ = -3k^2\delta_\nu^\mu + \text{diag}(\tilde{u}^\pm, \tilde{v}^\pm, \tilde{w}^\pm, \tilde{w}^\pm). \end{aligned} \quad (51)$$

Then, eq. (2) with $\gamma = 0$ and eq. (3) at $\xi = \pm\delta$ imply the same equations as (7)–(11) in (A) with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{u}^\pm \mp 2k\tilde{a}, \tilde{v}^\pm \mp 2k\tilde{b}$ and $\tilde{w}^\pm \mp 2k\tilde{c}$ in the places of a, b, c, u, v and w , respectively. Therefore, we obtain the same results as in (A) above, though the geometrical interpretations are different.

(C) *Einstein Hilbert Brane* ($\hat{\gamma} \neq 0, \gamma \neq 0, \hat{\lambda} = \lambda = 0$) The brane dynamics (2) implies, instead of (7) in (A),

$$\begin{aligned} &[-(h^{-1})'/r + (1 - h^{-1})/r^2]a \\ &+ [-f'/rfh + (1 - h^{-1})/r^2]b \\ &+ 2[-((fh)^{-1/2}f')'/2(fh)^{1/2} - (f/h)'/2rf]c = 0, \end{aligned} \quad (52)$$

while eqs. (8)–(11) are unchanged. If we eliminate f, f', f'', b, u , and w from eqs. (8)–(11) and (52), we obtain

$$\begin{aligned} (rh^{-1})' &= 1 + r^2V \\ &+ \frac{r^2[rV'(V - 2ac - 4c^2) + 5V^2 - 2Va(2a + c)]}{h(1 + r^2V)(V - 2ac - 4c^2) + 2(a + 2c)(2a + c)}. \end{aligned} \quad (53)$$

Eq. (53) has the same structure as (12) as far as h is concerned. Hence, we can apply the same method as that in (A) above. In the present case ($\lambda = 0$), the condition (47) automatically implicates the brane dynamics (52). Therefore, (47) just gives the condition for the brane Einstein equation to hold, but not (48).

In any cases, the successful results of the Einstein gravity are reproduced through fine tuning, and are not “predicted” by the braneworld theories. It would be an urgent and important challenge to seek for physical foundations of the fine-tuning conditions.

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